ORDER STATISTICS OF PEAKS IN EARTHQUAKE RESPONSE

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ABSTRACT: In its present form the response spectrum superposition technique provides only the highest peak of the response at various levels of a multistory structure. For better understanding of the progressing damage as the structure is subjected to successive excursions beyond the design level, and to estimate the number of times certain responses may be exceeded, it is essential to know all the peaks of the response, not just the highest peak. In this paper, a probabilistic theory is presented, using order statistics, to find the expected, the most probable, or with any desired confidence level, the amplitudes of all the local maxima in the random response functions at any point in a multistory structure.

INTRODUCTION

Many investigators have applied and extended the work of Rice (1944, 1945), Longuet-Higgins (1952) and Cartwright and Longuet-Higgins (1956) on the theory of probability distributions of a random function to the field of earthquake engineering and strong motion seismology. Udwadia and Trifunac (1974) studied the response spectra using the statistics of the largest peak of the response of a single-degree-of-freedom system.

Many studies have been carried out on the response of structures from a stochastic viewpoint. A few examples are those by Amini and Trifunac (1981, 1985), Balasubramonian and Iyer (1977), Bycroft (1960), Caughey and Gray (1963), Davenport (1964), DerKiureghian (1979, 1980), Gasparini (1979), Gasparini and Deb Chaudhury (1980), Grigoriu (1981), Gupta and Trifunac (1987a, b), Hammond (1968), Rosenblueth (1956, 1964), Rosenblueth and Bustamante (1962), Ruiz and Penzien (1971), Singh and Chu (1976), Tajimi (1960), Vanmarcke (1975), and Yang et al. (1980).

Amini and Trifunac (1981, 1985) extended the theory of Cartwright and Longuet-Higgins (1956) for the largest peak of a random function to study the statistics of higher-order peaks in the response of a structure under strong-motion earthquake excitation. To understand these response characteristics, which depend on the duration of the excitation, knowledge of all the peaks above certain threshold level is essential. For example, isolated in space and infrequent excursions in time of the response of ductile materials "not too far" into the nonlinear response range and beyond the linear design amplitudes are of particular interest for understanding the early phases of nonlinear and damaging response. For this purpose, the knowledge of the amplitudes of the first, second, third, etc., peaks of response is invaluable for estimating the number of times a certain response level will be exceeded. While ductile structures may experience

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repeated excursions beyond their elastic design amplitudes, without failing, the brittle structures might tolerate only one or two such excursions. Thus, to understand the onset of the damaging response it is essential to understand the relative peak amplitudes and the total number of the peaks of response to transient excitations.

Assuming that the history of response is a stationary process in time and that the peaks are statistically independent, Amini and Trifunac (1981, 1985) presented the results on the expected and the most probable amplitudes of higher-order peaks. Their theoretical results show good agreement with the first two or three highest peaks of computed response, and become poorer for higher-order peaks. Amini and Trifunac (1981, 1985) did not use the order statistics to derive the distribution functions for amplitudes of the higher-order peaks.

In this study, order statistics is applied to refine the theoretical distribution function for the amplitudes of n-th-order peak in a total of N peaks of a random function $f(t)$. Using this, relations are developed to compute the most probable and the expected amplitudes of the n-th-order peak. The use of order statistics has resulted in excellent improvement in theoretical predictions. The results on the expected and on the most probable peak amplitudes now show very good agreement with the trend of calculated response for all orders of peaks, even though the assumptions of stationarity and mutual independence of peaks have still been carried on. Improvements in the theory presented here over the earlier results by Amini and Trifunac (1981, 1985) are also confirmed by the ability of the present theory to predict negative peak amplitudes in accordance with the observations, as discussed later in this paper.

Most probable and expected values of the peaks are found to have about 50% probability of exceedance. The results are also presented on the values of peak amplitudes for the various probabilities of exceedance. These may be useful in determining the peak amplitudes with a desired degree of confidence.

Statistical Distribution of Maxima of Random Function

Following the work of Rice (1944, 1945) and Cartwright and Longuet-Higgins (1952, 1956), a random function of time, $f(t)$, which may represent the response of a structure to an earthquake excitation, can be represented by

$$f(t) = \sum_{n} C_n \cos(\omega_n t + \phi_n) \qquad (1)$$

where ω_n are the circular frequencies, ϕ_n are the random phases uniformly distributed between 0 and 2π, and C_n are the amplitudes related to the energy spectrum of $f(t)$ by the following relation

$$\sum_{\omega_n = \omega}^{\omega + d\omega} \frac{1}{2} C_n^2 = E(\omega)\, d\omega \qquad (2)$$

In the previous equation, $E(\omega)$ is the energy spectrum of $f(t)$.

Using previous definitions, Cartwright and Longuet-Higgins (1956) have derived the probability density function for the distribution of the maxima of $f(t)$, which depends only on two parameters: the root-mean-square (rms) value of $f(t)$, a_{rms} ; and a parameter ε, which is a measure of the width of the energy spectrum $E(\omega)$. These parameters are defined in terms of the moments of the energy spectrum as follows

$$a_{rms} = m_0^{1/2} \quad \text{(3)}$$

and

$$\varepsilon = \frac{m_0 m_4 - m_2^2}{m_0 m_4} \quad \text{(4)}$$

where, in general, the n-th moment, m_n , of the energy spectrum is defined by

$$m_n = \int_0^\infty \omega^n E(\omega)\, d\omega \qquad (n = 0, 1, 2, 3, \ldots) \quad \text{(5)}$$

The probability density function of the maxima of $f(t)$ normalized with respect to rms value $m_0^{1/2}$, is given by Cartwright and Lonquet-Higgins (1956) as

$$p(\eta) = \frac{1}{\sqrt{2\pi}} \left[\varepsilon e^{-1/2(\eta^2/\varepsilon^2)} + (1 - \varepsilon^2)^{1/2} \eta e^{-1/2\eta^2} \int_{-\infty}^{\frac{\eta(1-\varepsilon^2)^{1/2}}{\varepsilon}} e^{-1/2x^2}\, dx \right] \quad \text{(6)}$$

For $\varepsilon = 0$, this becomes Rayleigh distribution and for $\varepsilon = 1$ it becomes Gaussian distribution.

The cumulative probability that the height of a maximum will be greater than η can be defined as

$$P(\eta) = \int_\eta^\infty p(u)\, du \quad \text{(7)}$$

Using Eq. 6 for $p(u)$, Eq. 7 gives

$$P(\eta) = \frac{1}{\sqrt{2\pi}} \left[\int_{\eta/\varepsilon}^\infty e^{-x^2/2}\, dx + (1 - \varepsilon^2)^{1/2} e^{-\eta^2/2} \int_{-\infty}^{\frac{\eta(1-\varepsilon^2)^{1/2}}{\varepsilon}} e^{-x^2/2}\, dx \right] \quad \text{(8)}$$

The integrals in Eqs. 6 and 8 can be expressed in terms of error function

$$erf(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^x e^{-t^2/2}\, dt \quad \text{for} \quad x \geq 0 \quad \text{(9)}$$

and

$$erf(x) = -\,erf(-x) \quad \text{for} \quad x < 0 \quad \text{(10)}$$

Thus, one can write

$$p(\eta) = \frac{\varepsilon}{(2\pi)^{1/2}} e^{-1/2(\eta^2/\varepsilon^2)} + (1 - \varepsilon^2)^{1/2} \eta e^{-1/2\eta^2 x} \frac{1}{2} \left\{ 1 + erf\left[\frac{\eta(1-\varepsilon^2)^{1/2}}{\sqrt{2}\varepsilon} \right] \right\} \quad \text{for} \quad \eta \geq 0 \quad \text{(11a)}$$

$$p(\eta) = \frac{\varepsilon}{(2\pi)^{1/2}} e^{-1/2(\eta^2/\varepsilon^2)} + (1 - \varepsilon^2)^{1/2} \eta e^{-1/2\eta^2 x} \frac{1}{2} \left\{ 1 - erf\left[\frac{-\eta(1-\varepsilon^2)^{1/2}}{\sqrt{2}\varepsilon} \right] \right\} \quad \text{for} \quad \eta < 0 \quad \text{(11b)}$$

and

$$P(\eta)=\frac{1}{2}\left[1-erf\left(\frac{\eta}{\sqrt{2}\varepsilon}\right)\right]+\frac{1}{2}(1-\varepsilon^2)^{1/2}e^{-1/2\eta^2}\left\{1+erf\left[\frac{\eta(1-\varepsilon^2)^{1/2}}{\sqrt{2}\varepsilon}\right]\right\} \quad \text{for} \quad \eta\geq 0 \quad \ldots\ldots (12a)$$

$$P(\eta)=\frac{1}{2}\left[1+erf\left(\frac{-\eta}{\sqrt{2}\varepsilon}\right)\right]+\frac{1}{2}(1-\varepsilon^2)^{1/2}e^{-1/2\eta^2}\left\{1-erf\left[\frac{-\eta(1-\varepsilon^2)^{1/2}}{\sqrt{2}\varepsilon}\right]\right\} \quad \text{for} \quad \eta<0 \qquad (12b)$$

To evaluate $p(\eta)$ and $P(\eta)$ for particular values of η and ε the following approximation for $erf(x)$ (Hastings 1955) can be used

$$erf(x) = 1 - (a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5)e^{-x^2} + \varepsilon(x) \quad \ldots\ldots\ldots\ldots (13a)$$

$$|\varepsilon(x)| \leq 1.5 \times 10^{-7} \quad \ldots\ldots\ldots\ldots (13b)$$

where $t = 1/(1 + px)$, with $p = 0.3275911$, and $a_1 = 0.254829592$; $a_2 = -0.284496736$; $a_3 = 1.421413741$; $a_4 = -1.453152027$; and $a_5 = 1.061405429$.

Probability Distribution of n-th Local Maximum

One can represent the N peaks of a random function $f(t)$ by random variates $a_1, a_2, \ldots, a_N$; each with cumulative distribution function $P(\eta)$ given by

$$P(\eta) = \text{Prob}\{a_i > \eta\} = \int_{\eta}^{\infty} p(u)\, du \quad \ldots\ldots\ldots\ldots (14)$$

where $p(u)$ is the probability density function of the peaks of $f(t)$. The probability functions for the peaks of $f(t)$ normalized with respect to its rms amplitude are given by Eqs. 6 and 8. Let the peaks be now arranged in decreasing order of magnitudes as shown in Fig. 1. The new random variates are denoted by $a_{(1)}, a_{(2)}, a_{(3)}, \ldots, a_{(N)}$; where $a_{(1)} \geq a_{(2)} \geq a_{(3)}, \ldots, \geq a_{(N)}$. The probability that the n-th-order peak will exceed a value η can be written as

$$F_{(n)}(\eta) = \text{Prob}\{a_{(n)} > \eta\} \quad \ldots\ldots\ldots\ldots (15)$$

Because the peaks are arranged in decreasing order of magnitudes, the previous probability is equal to the probability that at least n of the peaks are greater than η. Therefore

$$F_{(n)}(\eta) = \text{Prob}\{\text{at least} \quad n \quad \text{of} \quad a_{(i)} \quad \text{are greater than} \quad \eta\} \quad \ldots\ldots (16)$$

It can also be written as

$$\begin{aligned} F_{(n)}(\eta) = \text{Prob}\{&\text{exactly} \quad n \quad \text{peaks are greater than} \quad \eta \quad \text{or} \\ &\text{exactly} \quad n+1 \quad \text{peaks are greater than} \quad \eta \quad \text{or} \\ &\text{exactly} \quad n+2 \quad \text{peaks are greater than} \quad \eta \quad \text{or} \\ &\vdots \\ &\text{exactly} \quad N \quad \text{peaks are greater than} \quad \eta\} \quad \ldots\ldots\ldots\ldots (17) \end{aligned}$$

Assuming that the peaks are statistically independent, the above probability can be rewritten as

$$
\begin{aligned}
F_{(n)}(\eta) = {} & \text{Prob}\{\text{exactly} \quad n \quad \text{peaks are greater than} \quad \eta\} \\
& + \text{Prob}\{\text{exactly} \quad n+1 \quad \text{peaks are greater than} \quad \eta\} \\
& + \text{Prob}\{\text{exactly} \quad n+2 \quad \text{peaks are greater than} \quad \eta\} \\
& \qquad \vdots \\
& + \text{Prob}\{\text{exactly} \quad N \quad \text{peaks are greater than} \quad \eta\} \quad \ldots\ldots\ldots \quad (18)
\end{aligned}
$$

TABLE 1. Amplitudes of First Ten Peaks Normalized by rms Amplitude for Different Values of N and Epsilon with Probability of Occurrence P = 0.01

	Order of Peak									
N (1)	First (2)	Second (3)	Third (4)	Fourth (5)	Fifth (6)	Sixth (7)	Seventh (8)	Eighth (9)	Ninth (10)	Tenth (11)
					(*a*) Epsilon = 0.0					
4	2.447	1.781	1.401	1.074	—	—	—	—	—	—
6	2.529	1.903	1.572	1.326	1.107	0.877	—	—	—	—
8	2.585	1.983	1.674	1.454	1.272	1.109	0.946	0.760	—	—
10	2.628	2.042	1.747	1.542	1.377	1.234	1.103	0.975	0.839	0.680
50	2.918	2.411	2.176	2.022	1.908	1.819	1.742	1.676	1.618	1.565
100	3.035	2.552	2.330	2.189	2.085	2.005	1.936	1.876	1.827	1.781
					(*b*) Epsilon = 0.2					
4	2.443	1.775	1.394	1.064	—	—	—	—	—	—
6	2.525	1.898	1.566	1.318	1.097	0.866	—	—	—	—
8	2.581	1.977	1.668	1.447	1.265	1.099	0.935	0.746	—	—
10	2.624	2.037	1.741	1.535	1.370	1.227	1.093	0.963	0.827	0.665
50	2.915	2.407	2.171	2.016	1.903	1.812	1.735	1.670	1.612	1.556
100	3.031	2.547	2.327	2.185	2.081	2.000	1.931	1.871	1.822	1.774
					(*c*) Epsilon = 0.4					
4	2.430	1.757	1.369	1.032	—	—	—	—	—	—
6	2.511	1.880	1.544	1.292	1.067	0.826	—	—	—	—
8	2.568	1.961	1.648	1.424	1.237	1.069	0.898	0.700	—	—
10	2.611	2.020	1.722	1.513	1.344	1.198	1.062	0.928	0.785	0.613
50	2.903	2.393	2.154	2.001	1.886	1.794	1.717	1.650	1.589	1.537
100	3.021	2.535	2.313	2.170	2.066	1.981	1.912	1.855	1.801	1.756
					(*d*) Epsilon = 0.6					
4	2.401	1.718	1.319	0.965	—	—	—	—	—	—
6	2.484	1.844	1.500	1.237	1.002	0.742	—	—	—	—
8	2.541	1.926	1.606	1.375	1.183	1.004	0.821	0.604	—	—
10	2.585	1.986	1.682	1.468	1.294	1.141	0.997	0.853	0.699	0.507
50	2.880	2.364	2.122	1.967	1.850	1.756	1.677	1.608	1.547	1.491
100	2.998	2.507	2.283	2.139	2.033	1.946	1.876	1.818	1.763	1.717
					(*e*) Epsilon = 0.8					
4	2.341	1.633	1.212	0.829	—	—	—	—	—	—
6	2.426	1.765	1.404	1.125	0.869	0.584	—	—	—	—
8	2.484	1.850	1.516	1.271	1.065	0.872	0.670	0.428	—	—
10	2.529	1.912	1.595	1.370	1.185	1.020	0.865	0.705	0.533	0.316
50	2.829	2.303	2.054	1.893	1.770	1.673	1.589	1.517	1.452	1.396
100	2.949	2.450	2.219	2.071	1.961	1.872	1.799	1.736	1.682	1.631
					(*f*) Epsilon = 1.0					
4	1.984	1.223	0.762	0.339	—	—	—	—	—	—
6	2.075	1.366	0.973	0.667	0.383	0.064	—	—	—	—
8	2.138	1.458	1.095	0.829	0.600	0.385	0.162	−0.110	—	—
10	2.185	1.525	1.182	0.935	0.733	0.551	0.378	0.202	0.009	−0.235
50	2.503	1.943	1.678	1.504	1.372	1.265	1.175	1.096	1.025	0.963
100	2.630	2.101	1.854	1.694	1.577	1.481	1.404	1.335	1.273	1.222

or

$$F_{(n)}(\eta) = \sum_{i=n}^{N} \text{Prob}\{\text{exactly } i \text{ out of } N \text{ peaks are greater than } \eta\} \quad \ldots\ldots\ldots \quad (19)$$

The probability that any one of the peaks is greater than η is given by $P(\eta)$ and is a constant for fixed values of m_0 and ε. Further, there are only

TABLE 2. Amplitudes of First Ten Peaks Normalized by rms Amplitude for Different Values of N and Epsilon with Probability of Occurrence $P = 0.10$

	Order of Peak									
N (1)	First (2)	Second (3)	Third (4)	Fourth (5)	Fifth (6)	Sixth (7)	Seventh (8)	Eighth (9)	Ninth (10)	Tenth (11)
					(*a*) Epsilon = 0.0					
4	1.911	1.396	1.067	0.759	—	—	—	—	—	—
6	2.013	1.543	1.267	1.049	0.845	0.620	—	—	—	—
8	2.083	1.637	1.386	1.196	1.033	0.879	0.723	0.537	—	—
10	2.136	1.706	1.469	1.295	1.150	1.019	0.897	0.773	0.641	0.481
50	2.483	2.131	1.952	1.829	1.734	1.657	1.591	1.534	1.481	1.435
100	2.619	2.289	2.122	2.011	1.927	1.858	1.799	1.750	1.704	1.662
					(*b*) Epsilon = 0.2					
4	1.905	1.389	1.057	0.745	—	—	—	—	—	—
6	2.008	1.537	1.259	1.039	0.834	0.603	—	—	—	—
8	2.078	1.631	1.378	1.188	1.022	0.868	0.708	0.518	—	—
10	2.131	1.700	1.462	1.287	1.140	1.010	0.884	0.761	0.626	0.458
50	2.479	2.126	1.946	1.824	1.729	1.652	1.585	1.527	1.475	1.428
100	2.615	2.284	2.118	2.006	1.922	1.853	1.794	1.744	1.697	1.657
					(*c*) Epsilon = 0.4					
4	1.888	1.365	1.025	0.700	—	—	—	—	—	—
6	1.991	1.514	1.232	1.007	0.793	0.546	—	—	—	—
8	2.062	1.611	1.354	1.159	0.989	0.829	0.660	0.451	—	—
10	2.115	1.681	1.439	1.260	1.111	0.976	0.846	0.715	0.570	0.384
50	2.466	2.111	1.929	1.804	1.710	1.631	1.564	1.506	1.451	1.404
100	2.602	2.269	2.103	1.990	1.904	1.834	1.775	1.724	1.679	1.637
					(*d*) Epsilon = 0.6					
4	1.852	1.314	0.958	0.604	—	—	—	—	—	—
6	1.957	1.469	1.176	0.938	0.706	0.431	—	—	—	—
8	2.029	1.568	1.303	1.099	0.919	0.746	0.559	0.321	—	—
10	2.082	1.640	1.391	1.205	1.048	0.905	0.766	0.621	0.458	0.242
50	2.437	2.078	1.894	1.766	1.669	1.588	1.519	1.459	1.405	1.355
100	2.576	2.239	2.070	1.955	1.868	1.797	1.736	1.684	1.638	1.594
					(*e*) Epsilon = 0.8					
4	1.773	1.206	0.821	0.427	—	—	—	—	—	—
6	1.882	1.371	1.058	0.799	0.543	0.231	—	—	—	—
8	1.957	1.476	1.194	0.975	0.779	0.587	0.377	0.104	—	—
10	2.012	1.551	1.289	1.090	0.919	0.763	0.608	0.447	0.263	0.013
50	2.378	2.008	1.817	1.685	1.583	1.498	1.426	1.361	1.303	1.250
100	2.520	2.175	2.000	1.881	1.790	1.716	1.653	1.598	1.549	1.505
					(*f*) Epsilon = 1.0					
4	1.374	0.756	0.330	−0.111	—	—	—	—	—	—
6	1.493	0.937	0.593	0.305	0.019	−0.333	—	—	—	—
8	1.573	1.051	0.742	0.501	0.283	0.068	−0.167	−0.476	—	—
10	1.633	1.133	0.846	0.628	0.440	0.265	0.093	−0.089	−0.297	−0.580
50	2.024	1.627	1.422	1.279	1.167	1.076	0.996	0.926	0.862	0.804
100	2.175	1.806	1.619	1.490	1.393	1.312	1.244	1.185	1.130	1.083

two possibilities: either a peak is greater than a particular value of η or it is less than η. Also, the peaks are assumed to be statistically independent. Thus the occurrence of peaks forms a Bernoulli sequence and the probability that exactly i out of N peaks will exceed a value η is given by binomial distribution (Ang and Tang 1975)

TABLE 3. Amplitudes of First Ten Peaks Normalized by rms Amplitude for Different Values of N and Epsilon with Probability of Occurrence P = 0.30

	Order of Peak									
N (1)	First (2)	Second (3)	Third (4)	Fourth (5)	Fifth (6)	Sixth (7)	Seventh (8)	Eighth (9)	Ninth (10)	Tenth (11)
					(*a*) Epsilon = 0.0					
4	1.569	1.141	0.843	0.549	—	—	—	—	—	—
6	1.689	1.306	1.063	0.862	0.669	0.448	—	—	—	—
8	1.770	1.411	1.193	1.021	0.869	0.724	0.572	0.388	—	—
10	1.831	1.488	1.284	1.127	0.994	0.872	0.755	0.636	0.508	0.347
50	2.224	1.955	1.806	1.701	1.618	1.549	1.489	1.437	1.389	1.344
100	2.375	2.125	1.989	1.895	1.821	1.760	1.708	1.661	1.620	1.583
					(*b*) Epsilon = 0.2					
4	1.563	1.131	0.831	0.530	—	—	—	—	—	—
6	1.683	1.298	1.053	0.850	0.653	0.425	—	—	—	—
8	1.764	1.404	1.184	1.011	0.857	0.709	0.554	0.361	—	—
10	1.825	1.481	1.276	1.118	0.984	0.861	0.741	0.621	0.487	0.316
50	2.220	1.950	1.801	1.695	1.612	1.543	1.482	1.430	1.381	1.337
100	2.370	2.120	1.984	1.889	1.815	1.754	1.701	1.655	1.614	1.577
					(*c*) Epsilon = 0.4					
4	1.541	1.102	0.790	0.465	—	—	—	—	—	—
6	1.663	1.272	1.021	0.810	0.600	0.344	—	—	—	—
8	1.745	1.380	1.155	0.977	0.817	0.661	0.491	0.269	—	—
10	1.807	1.459	1.249	1.088	0.949	0.821	0.695	0.564	0.416	0.214
50	2.205	1.933	1.782	1.676	1.591	1.521	1.460	1.406	1.357	1.312
100	2.356	2.104	1.967	1.871	1.796	1.734	1.682	1.635	1.593	1.555
					(*d*) Epsilon = 0.6					
4	1.496	1.039	0.703	0.337	—	—	—	—	—	—
6	1.622	1.218	0.953	0.725	0.493	0.196	—	—	—	—
8	1.706	1.330	1.095	0.906	0.734	0.561	0.368	0.104	—	—
10	1.769	1.411	1.194	1.024	0.876	0.737	0.599	0.452	0.281	0.038
50	2.174	1.897	1.744	1.634	1.548	1.476	1.412	1.357	1.306	1.259
100	2.327	2.072	1.933	1.834	1.758	1.695	1.641	1.593	1.550	1.511
					(*e*) Epsilon = 0.8					
4	1.400	0.910	0.539	0.123	—	—	—	—	—	—
6	1.533	1.103	0.816	0.564	0.302	−0.041	—	—	—	—
8	1.621	1.224	0.971	0.765	0.573	0.379	0.159	−0.148	—	—
10	1.687	1.310	1.078	0.894	0.732	0.578	0.422	0.255	0.058	−0.227
50	2.107	1.820	1.660	1.546	1.455	1.378	1.311	1.252	1.198	1.148
100	2.265	2.001	1.857	1.756	1.676	1.610	1.552	1.503	1.457	1.416
					(*f*) Epsilon = 1.0					
4	0.969	0.428	0.015	−0.455	—	—	—	—	—	—
6	1.113	0.643	0.324	0.043	−0.253	−0.642	—	—	—	—
8	1.210	0.775	0.496	0.267	0.053	−0.165	−0.415	−0.765	—	—
10	1.281	0.870	0.614	0.411	0.230	0.058	−0.116	−0.305	−0.530	−0.854
50	1.734	1.426	1.252	1.128	1.028	0.945	0.872	0.806	0.747	0.692
100	1.903	1.621	1.465	1.355	1.269	1.197	1.135	1.081	1.031	0.986

Prob{exactly i out of N peaks are greater than η}

$$= \binom{N}{i}[P(\eta)]^i[1 - P(\eta)]^{N-i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

Using Eq. 20 in Eq. 19 we get

TABLE 4. Amplitudes of First Ten Peaks Normalized by rms Amplitude for Different Values of N and Epsilon with Probability of Occurrence P = 0.50

	Order of Peak									
N (1)	First (2)	Second (3)	Third (4)	Fourth (5)	Fifth (6)	Sixth (7)	Seventh (8)	Eighth (9)	Ninth (10)	Tenth (11)
					(*a*) Epsilon = 0.0					
4	1.356	0.976	0.698	0.416	—	—	—	—	—	—
6	1.488	1.153	0.930	0.740	0.544	0.340	—	—	—	—
8	1.578	1.266	1.067	0.906	0.762	0.622	0.474	0.295	—	—
10	1.644	1.349	1.163	1.018	0.892	0.775	0.662	0.547	0.421	0.264
50	2.070	1.844	1.713	1.618	1.542	1.478	1.422	1.372	1.326	1.285
100	2.231	2.023	1.904	1.819	1.751	1.695	1.647	1.604	1.565	1.530
					(*b*) Epsilon = 0.2					
4	1.348	0.966	0.683	0.391	—	—	—	—	—	—
6	1.482	1.144	0.919	0.726	0.536	0.309	—	—	—	—
8	1.571	1.258	1.057	0.895	0.748	0.605	0.452	0.258	—	—
10	1.638	1.341	1.154	1.008	0.880	0.762	0.647	0.528	0.396	0.222
50	2.065	1.839	1.707	1.612	1.535	1.471	1.414	1.364	1.319	1.277
100	2.226	2.018	1.899	1.813	1.746	1.689	1.640	1.597	1.558	1.523
					(*c*) Epsilon = 0.4					
4	1.323	0.930	0.633	0.305	—	—	—	—	—	—
6	1.459	1.115	0.881	0.678	0.471	0.204	—	—	—	—
8	1.550	1.232	1.025	0.856	0.702	0.548	0.376	0.140	—	—
10	1.618	1.316	1.125	0.974	0.841	0.717	0.593	0.463	0.310	0.093
50	2.049	1.821	1.688	1.591	1.513	1.448	1.391	1.340	1.293	1.250
100	2.211	2.001	1.881	1.795	1.726	1.669	1.620	1.576	1.537	1.501
					(*d*) Epsilon = 0.6					
4	1.271	0.856	0.530	0.148	—	—	—	—	—	—
6	1.412	1.053	0.803	0.581	0.345	0.026	—	—	—	—
8	1.505	1.175	0.957	0.776	0.607	0.433	0.233	−0.054	—	—
10	1.575	1.263	1.063	0.903	0.760	0.623	0.485	0.335	0.155	−0.112
50	2.016	1.783	1.647	1.548	1.468	1.400	1.341	1.288	1.239	1.195
100	2.180	1.967	1.845	1.756	1.686	1.628	1.577	1.533	1.492	1.455
					(*e*) Epsilon = 0.8					
4	1.161	0.709	0.344	−0.097	—	—	—	—	—	—
6	1.311	0.925	0.651	0.401	0.132	−0.241	—	—	—	—
8	1.410	1.058	0.821	0.621	0.431	0.234	0.002	−0.336	—	—
10	1.483	1.152	0.936	0.761	0.602	0.450	0.293	0.120	−0.089	−0.405
50	1.943	1.701	1.559	1.455	1.370	1.298	1.235	1.179	1.127	1.079
100	2.113	1.893	1.766	1.674	1.601	1.539	1.486	1.439	1.396	1.357
					(*f*) Epsilon = 1.0					
4	0.706	0.205	−0.205	−0.706	—	—	—	—	—	—
6	0.871	0.445	0.140	−0.140	−0.445	−0.871	—	—	—	—
8	0.980	0.592	0.330	0.106	−0.106	−0.330	−0.592	−0.980	—	—
10	1.060	0.697	0.458	0.263	0.086	−0.086	−0.263	−0.458	−0.697	−1.060
50	1.558	1.297	1.142	1.028	0.936	0.857	0.788	0.726	0.668	0.615
100	1.741	1.504	1.367	1.267	1.187	1.121	1.063	1.011	0.964	0.921

$$F_{(n)}(\eta) = \sum_{i=n}^{N} \binom{N}{i} [P(\eta)]^i [1 - P(\eta)]^{N-i} \quad (21)$$

This gives the cumulative probability that the n-th-order peak in a total of N peaks of a random function $f(t)$ will exceed an amplitude η. Let $\bar{a}$ be the rms amplitude of the peaks, defined by

TABLE 5. Amplitudes of First Ten Peaks Normalized by rms Amplitude for Different Values of N and Epsilon with Probability of Occurrence $P = 0.70$

	Order of Peak									
N (1)	First (2)	Second (3)	Third (4)	Fourth (5)	Fifth (6)	Sixth (7)	Seventh (8)	Eighth (9)	Ninth (10)	Tenth (11)
(a) Epsilon = 0.0										
4	1.161	0.822	0.564	0.298	—	—	—	—	—	—
6	1.306	1.010	0.804	0.625	0.448	0.244	—	—	—	—
8	1.403	1.130	0.947	0.797	0.660	0.525	0.383	0.211	—	—
10	1.475	1.218	1.048	0.913	0.793	0.682	0.574	0.462	0.340	0.189
50	1.933	1.742	1.626	1.539	1.468	1.409	1.356	1.309	1.266	1.226
100	2.104	1.929	1.824	1.747	1.685	1.633	1.588	1.547	1.511	1.478
(b) Epsilon = 0.2										
4	1.152	0.810	0.545	0.262	—	—	—	—	—	—
6	1.298	1.000	0.791	0.608	0.425	0.199	—	—	—	—
8	1.395	1.121	0.936	0.784	0.644	0.505	0.355	0.159	—	—
10	1.468	1.209	1.039	0.901	0.781	0.667	0.556	0.440	0.309	0.131
50	1.928	1.736	1.619	1.532	1.462	1.401	1.349	1.301	1.258	1.217
100	2.099	1.923	1.818	1.741	1.679	1.627	1.581	1.541	1.504	1.470
(c) Epsilon = 0.4										
4	1.122	0.768	0.482	0.146	—	—	—	—	—	—
6	1.272	0.966	0.748	0.551	0.344	0.062	—	—	—	—
8	1.371	1.091	0.900	0.740	0.590	0.437	0.262	0.008	—	—
10	1.445	1.181	1.006	0.864	0.737	0.616	0.494	0.362	0.204	−0.032
50	1.911	1.717	1.598	1.510	1.439	1.378	1.324	1.276	1.231	1.190
100	2.083	1.906	1.800	1.722	1.659	1.606	1.560	1.519	1.482	1.448
(d) Epsilon = 0.6										
4	1.061	0.679	0.357	−0.046	—	—	—	—	—	—
6	1.217	0.894	0.657	0.437	0.195	−0.151	—	—	—	—
8	1.321	1.028	0.824	0.649	0.481	0.305	0.096	−0.220	—	—
10	1.398	1.123	0.937	0.784	0.645	0.510	0.371	0.217	0.026	−0.271
50	1.875	1.677	1.556	1.465	1.391	1.327	1.272	1.221	1.175	1.132
100	2.050	1.870	1.762	1.682	1.618	1.563	1.516	1.474	1.435	1.400
(e) Epsilon = 0.8										
4	0.934	0.512	0.147	−0.327	—	—	—	—	—	—
6	1.103	0.752	0.488	0.238	−0.042	−0.452	—	—	—	—
8	1.214	0.897	0.674	0.478	0.289	0.086	−0.158	−0.535	—	—
10	1.296	1.001	0.799	0.630	0.474	0.322	0.162	−0.017	−0.241	−0.597
50	1.797	1.590	1.463	1.367	1.288	1.221	1.161	1.107	1.057	1.010
100	1.979	1.792	1.679	1.596	1.529	1.471	1.421	1.377	1.336	1.298
(f) Epsilon = 1.0										
4	0.455	−0.015	−0.428	−0.969	—	—	—	—	—	—
6	0.642	0.253	−0.043	−0.324	−0.643	−1.113	—	—	—	—
8	0.765	0.415	0.165	−0.053	−0.267	−0.496	−0.775	−1.210	—	—
10	0.854	0.530	0.305	0.166	−0.058	−0.230	−0.411	−0.614	−0.870	−1.281
50	1.401	1.176	1.037	0.932	0.847	0.772	0.707	0.647	0.592	0.540
100	1.597	1.395	1.273	1.182	1.109	1.046	0.992	0.943	0.898	0.858

$$\bar{a} = \left[\frac{1}{N}(a_1^2 + a_2^2 + a_3^2 + \cdots + a_N^2)\right]^{1/2} \qquad (22)$$

For a narrow-band function $f(t)$ (i.e., for $\varepsilon \approx 0$), $\bar{a}$ can be approximated in terms of

TABLE 6. Amplitudes of First Ten Peaks Normalized by rms Amplitude for Different Values of N and Epsilon with Probability of Occurrence P = 0.90

	Order of Peak									
N (1)	First (2)	Second (3)	Third (4)	Fourth (5)	Fifth (6)	Sixth (7)	Seventh (8)	Eighth (9)	Ninth (10)	Tenth (11)
(*a*) Epsilon = 0.0										
4	0.909	0.621	0.392	0.162	—	—	—	—	—	—
6	1.069	0.820	0.637	0.473	0.311	0.132	—	—	—	—
8	1.177	0.949	0.786	0.649	0.523	0.398	0.266	0.114	—	—
10	1.257	1.042	0.893	0.771	0.660	0.557	0.455	0.350	0.236	0.102
50	1.760	1.606	1.507	1.430	1.368	1.314	1.265	1.221	1.181	1.144
100	1.945	1.806	1.716	1.649	1.595	1.548	1.506	1.469	1.435	1.405
(*b*) Epsilon = 0.2										
4	0.897	0.605	0.365	0.094	—	—	—	—	—	—
6	1.060	0.807	0.620	0.451	0.277	0.051	—	—	—	—
8	1.168	0.938	0.774	0.634	0.503	0.371	0.225	0.023	—	—
10	1.249	1.033	0.882	0.757	0.645	0.538	0.432	0.320	0.190	0.003
50	1.754	1.600	1.500	1.424	1.360	1.307	1.257	1.213	1.173	1.136
100	1.939	1.800	1.710	1.643	1.589	1.541	1.499	1.462	1.428	1.397
(*c*) Epsilon = 0.4										
4	0.859	0.547	0.273	−0.083	—	—	—	—	—	—
6	1.028	0.765	0.565	0.375	0.164	−0.147	—	—	—	—
8	1.139	0.901	0.729	0.579	0.434	0.282	0.098	−0.190	—	—
10	1.222	1.000	0.844	0.712	0.591	0.474	0.354	0.218	0.050	−0.222
50	1.736	1.579	1.478	1.400	1.336	1.280	1.231	1.185	1.144	1.105
100	1.922	1.781	1.691	1.624	1.567	1.521	1.478	1.440	1.405	1.374
(*d*) Epsilon = 0.6										
4	0.780	0.433	0.111	−0.338	—	—	—	—	—	—
6	0.960	0.676	0.452	0.232	−0.023	−0.421	—	—	—	—
8	1.078	0.825	0.637	0.469	0.302	0.120	−0.105	−0.477	—	—
10	1.166	0.931	0.762	0.618	0.483	0.349	0.206	0.044	−0.166	−0.519
50	1.696	1.535	1.431	1.351	1.284	1.226	1.174	1.128	1.083	1.042
100	1.886	1.742	1.650	1.580	1.524	1.474	1.430	1.391	1.356	1.323
(*e*) Epsilon = 0.8										
4	0.624	0.233	−0.141	−0.677	—	—	—	—	—	—
6	0.824	0.509	0.255	0.002	−0.299	−0.779	—	—	—	—
8	0.953	0.675	0.465	0.275	0.083	−0.130	−0.398	−0.848	—	—
10	1.047	0.792	0.605	0.443	0.291	0.137	−0.029	−0.220	−0.470	−0.900
50	1.610	1.442	1.331	1.246	1.175	1.112	1.057	1.005	0.959	0.915
100	1.809	1.660	1.563	1.490	1.429	1.378	1.331	1.289	1.251	1.216
(*f*) Epsilon = 1.0										
4	0.111	−0.330	−0.756	−1.374	—	—	—	—	—	—
6	0.333	−0.019	−0.305	−0.593	−0.937	−1.493	—	—	—	—
8	0.476	0.167	0.068	−0.283	−0.501	−0.742	−1.051	−1.573	—	—
10	0.580	0.297	−0.089	−0.093	−0.265	−0.440	−0.628	−0.846	−1.133	−1.633
50	1.199	1.014	0.894	0.800	0.721	0.652	0.591	0.535	0.483	0.434
100	1.414	1.251	1.146	1.067	1.000	1.943	0.893	0.849	0.807	0.768
1,000	2.003	1.881	1.806	1.748	1.703	1.665	1.631	1.600	1.573	1.549

$$a_{rms} = \left[\frac{1}{T} \int_0^T f(t)\, dt \right]^{1/2} \quad \text{. . . . .} \quad (23)$$

the rms value of $f(t)$, as follows (Udwadia and Trifunac 1974)

$$\bar{a} = \sqrt{2}\, a_{rms} = \sqrt{2}\, m_0^{1/2} \quad \text{. . . . .} \quad (24)$$

TABLE 7. Amplitudes of First Ten Peaks Normalized by rms Amplitude for Different Values of N and Epsilon with Probability of Occurrence P = 0.99

	ORDER OF PEAK									
N (1)	First (2)	Second (3)	Third (4)	Fourth (5)	Fifth (6)	Sixth (7)	Seventh (8)	Eighth (9)	Ninth (10)	Tenth (11)
					(*a*) Epsilon = 0.0					
4	0.615	0.390	0.206	0.047	—	—	—	—	—	—
6	0.789	0.589	0.434	0.296	0.162	0.039	—	—	—	—
8	0.908	0.725	0.587	0.468	0.358	0.250	0.141	0.035	—	—
10	0.997	0.826	0.700	0.592	0.496	0.401	0.311	0.219	0.122	0.028
50	1.558	1.439	1.357	1.292	1.239	1.189	1.146	1.107	1.070	1.036
100	1.760	1.654	1.582	1.526	1.479	1.437	1.400	1.367	1.339	1.311
					(*b*) Epsilon = 0.2					
4	0.600	0.361	0.153	−0.110	—	—	—	—	—	—
6	0.777	0.572	0.410	0.260	0.097	−0.137	—	—	—	—
8	0.896	0.711	0.571	0.466	0.328	0.206	0.063	−0.156	—	—
10	0.988	0.815	0.684	0.575	0.473	0.376	0.277	0.170	0.039	−0.170
50	1.552	1.431	1.350	1.284	1.230	1.180	1.138	1.099	1.063	1.028
100	1.753	1.647	1.575	1.521	1.471	1.429	1.392	1.359	1.329	1.303
					(*c*) Epsilon = 0.4					
4	0.541	0.269	0.001	−0.402	—	—	—	—	—	—
6	0.732	0.511	0.328	0.144	−0.079	−0.447	—	—	—	—
8	0.859	0.663	0.509	0.369	0.228	0.073	−0.129	−0.478	—	—
10	0.954	0.772	0.636	0.514	0.401	0.288	0.165	0.023	−0.166	−0.501
50	1.529	1.409	1.324	1.258	1.204	1.152	1.107	1.067	1.030	0.994
100	1.735	1.628	1.556	1.496	1.450	1.407	1.369	1.336	1.307	1.277
					(*d*) Epsilon = 0.6					
4	0.426	0.107	−0.230	−0.762	—	—	—	—	—	—
6	0.640	0.392	0.178	−0.049	−0.332	−0.824	—	—	—	—
8	0.780	0.564	0.391	0.226	0.056	−0.138	−0.397	−0.867	—	—
10	0.881	0.684	0.533	0.396	0.263	0.127	−0.022	−0.200	−0.445	−0.899
50	1.486	1.359	1.274	1.205	1.144	1.094	1.045	1.002	0.962	0.925
100	1.696	1.585	1.510	1.452	1.400	1.357	1.318	1.283	1.251	1.221
					(*e*) Epsilon = 0.8					
4	0.225	−0.146	−0.547	−1.200	—	—	—	—	—	—
6	0.469	0.186	−0.063	−0.330	−0.670	−1.278	—	—	—	—
8	0.624	0.382	0.184	−0.005	−0.204	−0.437	−0.750	−1.331	—	—
10	0.737	0.518	0.347	0.191	0.039	−0.121	−0.298	−0.511	−0.807	−1.371
50	1.388	1.255	1.164	1.089	1.026	0.968	0.920	0.870	0.826	0.785
100	1.609	1.494	1.416	1.352	1.300	1.253	1.212	1.175	1.140	1.107
					(*f*) Epsilon = 1.0					
4	−0.339	−0.762	−1.223	−1.984	—	—	—	—	—	—
6	−0.064	−0.383	−0.667	−0.973	−1.366	−2.075	—	—	—	—
8	0.110	−0.162	−0.385	−0.600	−0.829	−1.095	−1.458	−2.138	—	—
10	0.235	−0.009	−0.202	−0.378	−0.551	−0.733	−0.935	−1.182	−1.525	−2.185
50	0.957	0.811	0.709	0.626	0.556	0.496	0.437	0.387	0.336	0.290
100	1.198	1.071	0.986	0.918	0.859	0.810	0.763	0.721	0.683	0.648

In Eq. 23 T is the total duration of $f(t)$. The value of η obtained from Eq. 21 by using the distribution functions of Eq. 8 is the height of a peak divided by $m_0^{1/2}$. Therefore, $\eta/\sqrt{2}$ gives the height of a peak normalized by $\bar{a}$.

Computations have been made for the normalized amplitudes of the first 10 peaks for various values of ε and N. Tables 1 to 7 show these results for the probability of exceedance equal to 0.01, 0.1, 0.3, 0.5, 0.7, 0.9, and 0.99.

MOST PROBABLE VALUE OF n-TH ORDER PEAK

From Eq. 21, the probability density function for the n-th-order peak can be written as

$$f_{(n)}(\eta) = -\frac{dF_{(n)}(\eta)}{d\eta}$$

$$f_{(n)}(\eta) = -\sum_{i=n}^{N}\binom{N}{i}\{i[P(\eta)]^{i-1}[1-P(\eta)]^{N-i}$$

$$-(N-i)[P(\eta)]^{i}[1-P(\eta)]^{N-i-1}\}\frac{dP(\eta)}{d\eta} \quad (25)$$

Since $\frac{dP(\eta)}{d\eta} = -p(\eta)$, Eq. 25 can be written as

$$f_{(n)}(\eta) = \sum_{i=n}^{N}\binom{N}{i}\{i[P(\eta)]^{i-1}[1-P(\eta)]^{N-i} - (N-i)[P(\eta)]^{i}[1-P(\eta)]^{N-i-1}\}p(\eta) \quad (26)$$

Expanding the summation in the above equation gives

$$f_{(n)}(\eta) = \left\{\binom{N}{n}n[P(\eta)]^{n-1}[1-P(\eta)]^{N-n} - \binom{N}{n+1}(n+1)[P(\eta)]^{n}\right.$$

$$\left.\cdot[1-p(\eta)]^{N-n-1}\right\}p(\eta) + \left\{\binom{N}{n+1}(n+1)[P(\eta)]^{n}[1-P(\eta)]^{N-n-1}\right.$$

$$\left.-\binom{N}{n+2}(n+2)[P(\eta)]^{n+1}[1-P(\eta)]^{N-n-2}\right\}p(\eta) + \left\{\binom{N}{n+2}\right.$$

$$\cdot(n+2)[P(\eta)]^{n+1}[1-P(\eta)]^{N-n-2} - \binom{N}{n+3}$$

$$\left.\cdot(n+3)[P(\eta)]^{n+2}[1-P(\eta)]^{N-n-3}\right\}p(\eta)$$

$$\vdots$$

$$+\left\{\binom{N}{N}N[P(\eta)]^{N-1}[1-P(\eta)]^{N-N} - 0\right\}p(\eta) \quad (27)$$

In writing the above expansion, the following equality has been used

$$\binom{N}{i}(N-i) = \binom{N}{i+1}(i+1) \quad \text{for} \quad i = n, \quad n+1, \ldots, N \ldots\ldots\ldots (28)$$

Thus, the probability density function for the n-th-order peak becomes

$$f_{(n)}(\eta) = n\binom{N}{n}[P(\eta)]^{n-1}[1 - P(\eta)]^{N-n}p(\eta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

TABLE 8. Most Probable Peak Amplitudes Normalized by rms Amplitude for First Ten Peaks and for Different Values of N and Epsilon

	ORDER OF PEAK									
N (1)	First (2)	Second (3)	Third (4)	Fourth (5)	Fifth (6)	Sixth (7)	Seventh (8)	Eighth (9)	Ninth (10)	Tenth (11)
					(*a*) Epsilon = 0.0					
4	1.291	0.936	0.658	0.354	—	—	—	—	—	—
6	1.426	1.119	0.902	0.713	0.524	0.289	—	—	—	—
8	1.516	1.235	1.043	0.886	0.742	0.601	0.448	0.251	—	—
10	1.583	1.318	1.141	0.999	0.875	0.759	0.646	0.529	0.398	0.224
50	2.011	1.815	1.694	1.604	1.531	1.468	1.414	1.365	1.320	1.279
100	2.172	1.994	1.885	1.805	1.740	1.686	1.639	1.597	1.559	1.524
					(*b*) Epsilon = 0.2					
4	1.284	0.927	0.648	0.352	—	—	—	—	—	—
6	1.420	1.111	0.893	0.702	0.511	0.284	—	—	—	—
8	1.510	1.227	1.034	0.875	0.730	0.587	0.433	0.242	—	—
10	1.577	1.311	1.133	0.990	0.864	0.747	0.632	0.513	0.381	0.221
50	2.006	1.810	1.689	1.598	1.524	1.461	1.407	1.357	1.312	1.271
100	2.168	1.989	1.880	1.799	1.735	1.680	1.632	1.590	1.552	1.518
					(*c*) Epsilon = 0.4					
4	1.263	0.899	0.612	0.300	—	—	—	—	—	—
6	1.399	1.085	0.861	0.663	0.462	0.210	—	—	—	—
8	1.490	1.202	1.005	0.841	0.690	0.539	0.372	0.151	—	—
10	1.558	1.287	1.105	0.959	0.829	0.707	0.586	0.458	0.310	0.107
50	1.990	1.792	1.669	1.577	1.502	1.439	1.383	1.333	1.287	1.245
100	2.153	1.973	1.862	1.781	1.715	1.660	1.612	1.569	1.531	1.495
					(*d*) Epsilon = 0.6					
4	1.218	0.835	0.524	0.166	—	—	—	—	—	—
6	1.356	1.029	0.791	0.575	0.348	0.051	—	—	—	—
8	1.449	1.150	0.943	0.767	0.602	0.433	0.241	−0.024	—	—
10	1.518	1.237	1.048	0.892	0.753	0.619	0.484	0.338	0.165	−0.080
50	1.956	1.754	1.629	1.534	1.457	1.391	1.334	1.282	1.234	1.190
100	2.122	1.939	1.826	1.743	1.676	1.619	1.570	1.526	1.486	1.450
					(*e*) Epsilon = 0.8					
4	1.115	0.697	0.348	−0.063	—	—	—	—	—	—
6	1.260	0.906	0.644	0.403	0.143	−0.201	—	—	—	—
8	1.357	1.036	0.810	0.616	0.431	0.239	0.018	−0.293	—	—
10	1.429	1.129	0.924	0.754	0.599	0.449	0.296	0.129	−0.072	−0.360
50	1.884	1.674	1.541	1.442	1.360	1.290	1.229	1.173	1.122	1.074
100	2.055	1.865	1.747	1.661	1.590	1.531	1.479	1.432	1.390	1.352
					(*f*) Epsilon = 1.0					
4	0.662	0.196	−0.196	−0.662	—	—	—	—	—	—
6	0.821	0.428	0.136	−0.136	−0.428	−0.821	—	—	—	—
8	0.926	0.572	0.321	0.104	−0.104	−0.321	−0.572	−0.926	—	—
10	1.004	0.674	0.446	0.257	0.084	−0.084	−0.257	−0.446	−0.674	−1.004
50	1.497	1.268	1.124	1.015	0.926	0.849	0.782	0.720	0.664	0.611
100	1.679	1.475	1.348	1.253	1.176	1.112	1.005	1.004	0.958	0.916

The most probable value of n-th-order peak is the value of η for which $f_{(n)}(\eta)$ has the largest value. In order to find the most probable value $\mu[a_{(n)}]$, the function $f_{(n)}(\eta)$ has been evaluated for various values of η at closely spaced points with intervals equal to 0.05. By comparing the values of $f_n(\eta)$ at three consecutive points and by scanning the whole range of η, one can locate three points around the peak. A parabola fitted through these three points can locate the value of η for which the first derivative of the parabola vanishes. This value of η has been taken as the most probable value of the n-th-order peak. Table 8 lists the most probable amplitudes of the first 10 peaks for $\varepsilon = 0.0, 0.2, 0.4, 0.6, 0.8$, and 1.0 and for total number of peaks $N = 4, 6, 8, 10, 50$, and 100. Most probable values of the first-order peak are found to be in excellent agreement with the results of Amini and Trifunac (1981).

EXPECTED VALUE OF n-TH ORDER PEAK

Using the probability density function $f_{(n)}(\eta)$ given by Eq. 29, the expected (mean) value of n-th-order peak can be found from

$$E[a_{(n)}] = \int_{-\infty}^{\infty} \eta f_{(n)}(\eta)\, d\eta \qquad (30a)$$

$$E[a_{(n)}] = n\binom{N}{n} \int_{-\infty}^{\infty} \eta[P(\eta)]^{n-1}[1 - P(\eta)]^{N-n}p(\eta)\, d\eta \qquad (30b)$$

One can write $p(\eta)d\eta = -dP(\eta)$. Therefore

$$E[a_{(n)}] = -n\binom{N}{n} \int_{-\infty}^{\infty} \eta P^{n-1}(1 - P)^{N-n}\, dP \qquad (31)$$

Making the change of variable $X = P(\eta)$, Eq. 31 can be written as

$$E[a_{(n)}] = -n\binom{N}{n} \int_{1}^{0} P^{-1}(X)X^{n-1}(1 - X)^{N-n}\, dX \qquad (32a)$$

$$E[a_{(n)}] = n\binom{N}{n} \int_{0}^{1} P^{-1}(X)X^{n-1}(1 - X)^{N-n}\, dX \qquad (32b)$$

Expanding $X = P^{-1}[P(X)] \equiv Q[P(X)]$ in a Taylor series about $P_0 = n/(N + 1)$, where the choice of P_0 will be explained later in the paper, one can write

$$X = Q[P(X)] = Q(P_0) + [P(X) - P_0]Q'(P_0) + \frac{1}{2}[P(X) - P_0]^2Q''(P_0)$$
$$+ \frac{1}{6}[P(X) - P_0]^3Q'''(P_0) + \dots \qquad (33)$$

It has been shown by David (1980), using the expansion of Eq. 33, that the value of the integral in Eq. 32b can be approximated, to order $(N + 2)^{-2}$, by

$$E[a_{(n)}] = Q(P_0) + \frac{P_0(1-P_0)}{2(N+2)} Q''(P_0) + \frac{P_0(1-P_0)}{(N+2)^2}\left[\frac{1}{3}(1-2P_0)Q'''(P_0) + \frac{1}{8} P_0(1-P_0)Q''''(P_0)\right] \quad (34)$$

In Eq. 34

$$Q(P_0) = P^{-1}(P_0) \quad (35)$$

and

$$P_0 = \frac{n}{N+1} \quad (36)$$

P_0 is the expected value of the n-th-order statistic for the density function $p(x)$, uniform in (0, 1)

$$P_0 = n\binom{N}{n} \int_0^1 XX^{n-1}(1-X)^{N-n}\, dX \quad (37a)$$

$$P_0 = \frac{n\binom{N}{n}}{(N+1)\binom{N}{n}} = \frac{n}{N+1} \quad (37b)$$

Derivatives of the function $Q(P)$ in Eq. 34 can be found as follows

$$Q'(P) = \frac{dQ}{dP} = \frac{1}{\frac{dP}{dQ}} = \frac{1}{\frac{d}{dQ}[P(Q)]} = -\frac{1}{p(Q)} \quad (38)$$

$$Q''(P) = \frac{d[Q'(P)]}{dP} = \frac{d[Q'(P)]}{dQ} \cdot \frac{dQ}{dP} = \frac{d}{dQ}\left[-\frac{1}{p(Q)}\right] \frac{1}{\frac{dP(Q)}{dQ}} \quad (39a)$$

$$Q''(P) = \frac{1}{p(Q)} \cdot \frac{p'(Q)}{[p(Q)]^2} \quad (39b)$$

Using Eqs. 6 and 8 for $p(Q)$ and $P(Q)$ it can be seen that

$$Q''(P_0) = \frac{Q(P_0)}{p^2(Q)} - \frac{f(Q)}{p^3(Q)} \quad (40)$$

where

$$f(Q) = (1-\varepsilon^2)^{1/2} e^{-1/2Q^2} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{Q\frac{(1-\varepsilon^2)^{1/2}}{\varepsilon}} e^{-1/2x^2}\, dx \quad (41)$$

Next, it can be seen that

$$Q'''(P_0) = -\frac{2+2Q^2}{p^3(Q)} + \frac{g(Q) + 5Qf(Q)}{p^4(Q)} - \frac{3f^2(Q)}{p^5(Q)} \quad (42)$$

where $f(Q)$ is given by Eq. 41, and

$$g(Q) = \frac{1}{\varepsilon} \cdot \frac{1}{\sqrt{2\pi}} e^{-1/2(Q^2/\varepsilon^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (43)$$

Similarly,

TABLE 9. Expected Peak Amplitudes Normalized by rms Amplitude for First Ten Peaks and for Different Values of N and Epsilon

	ORDER OF PEAK									
N (1)	First (2)	Second (3)	Third (4)	Fourth (5)	Fifth (6)	Sixth (7)	Seventh (8)	Eighth (9)	Ninth (10)	Tenth (11)
					(*a*) Epsilon = 0.0					
4	1.385	0.994	0.717	0.442	—	—	—	—	—	—
6	1.518	1.169	0.943	0.752	0.569	0.360	—	—	—	—
8	1.609	1.282	1.078	0.916	0.771	0.632	0.486	0.312	—	—
10	1.676	1.364	1.174	1.026	0.900	0.783	0.670	0.556	0.432	0.279
50	2.103	1.859	1.723	1.625	1.548	1.483	1.426	1.375	1.330	1.287
100	2.264	2.038	1.914	1.826	1.757	1.700	1.651	1.607	1.568	1.532
					(*b*) Epsilon = 0.2					
4	1.377	0.983	0.700	0.409	—	—	—	—	—	—
6	1.512	1.160	0.931	0.737	0.547	0.319	—	—	—	—
8	1.602	1.274	1.068	0.904	0.757	0.614	0.461	0.264	—	—
10	1.670	1.356	1.165	1.016	0.888	0.769	0.654	0.535	0.403	0.226
50	2.099	1.854	1.718	1.619	1.541	1.476	1.419	1.368	1.322	1.280
100	2.260	2.033	1.909	1.821	1.751	1.694	1.644	1.601	1.562	1.526
					(*c*) Epsilon = 0.4					
4	1.350	0.945	0.643	0.308	—	—	—	—	—	—
6	1.488	1.129	0.891	0.686	0.476	0.201	—	—	—	—
8	1.580	1.246	1.035	0.864	0.708	0.552	0.378	0.134	—	—
10	1.648	1.330	1.135	0.981	0.847	0.722	0.597	0.465	0.310	0.085
50	2.084	1.836	1.698	1.598	1.519	1.453	1.395	1.343	1.296	1.253
100	2.245	2.017	1.891	1.802	1.732	1.674	1.624	1.580	1.540	1.504
					(*d*) Epsilon = 0.6					
4	1.294	0.865	0.533	0.141	—	—	—	—	—	—
6	1.438	1.064	0.810	0.583	0.343	0.014	—	—	—	—
8	1.534	1.187	0.965	0.781	0.609	0.433	0.230	−0.068	—	—
10	1.605	1.276	1.071	0.908	0.763	0.625	0.486	0.333	0.150	−0.129
50	2.050	1.798	1.657	1.555	1.473	1.405	1.345	1.291	1.242	1.197
100	2.214	1.982	1.855	1.763	1.692	1.633	1.582	1.536	1.495	1.458
					(*e*) Epsilon = 0.8					
4	1.180	0.715	0.342	−0.111	—	—	—	—	—	—
6	1.335	0.933	0.654	0.401	0.127	−0.260	—	—	—	—
8	1.437	1.068	0.826	0.623	0.431	0.231	−0.005	−0.358	—	—
10	1.512	1.164	0.943	0.764	0.604	0.450	0.291	0.116	−0.098	−0.429
50	1.977	1.716	1.569	1.462	1.376	1.303	1.239	1.182	1.129	1.081
100	2.148	1.908	1.775	1.681	1.606	1.544	1.490	1.442	1.399	1.359
					(*f*) Epsilon = 1.0					
4	0.724	0.209	−0.209	−0.724	—	—	—	—	—	—
6	0.895	0.453	0.142	−0.142	−0.453	−0.895	—	—	—	—
8	1.007	0.602	0.334	0.108	−0.108	−0.334	−0.602	−1.007	—	—
10	1.089	0.708	0.464	0.266	0.087	−0.087	−0.266	−0.464	−0.708	−1.089
50	1.594	1.312	1.152	1.035	0.941	0.862	0.792	0.729	0.671	0.618
100	1.777	1.520	1.377	1.274	1.193	1.125	1.067	1.014	0.967	0.924

$$Q''''(P_0) = \frac{Q(15 + 6Q^2)}{p^4(Q)} + \frac{h(Q) - 17f(Q) - 9Qg(Q) - 26Q^2f(Q)}{p^5(Q)}$$
$$+ \frac{35Qf^2(Q) + 10g(Q)f(Q)}{p^6(Q)} - \frac{15f^3(Q)}{p^7(Q)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (44)$$

where $f(Q)$ and $g(Q)$ are given by Eqs. 41 and 43, and

$$h(Q) = \frac{1}{\varepsilon^2} \cdot \frac{Q}{\sqrt{2\pi}} \quad \exp \quad \left(-\frac{1}{2}\frac{Q^2}{\varepsilon^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (45)$$

For $\varepsilon = 0$, the above expressions simplify to

$$Q(P_0) = \left(\ln \frac{1}{P_0^2}\right)^{1/2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (46)$$

$$p(Q) = P_0\left(\ln \frac{1}{P_0^2}\right)^{1/2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (47)$$

$$f(Q) = P_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (48)$$

and

$$g(Q) = h(Q) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (49)$$

Table 9 lists the expected values of the first ten peaks for $\varepsilon = 0.0$, 0.2, 0.4, 0.6, 0.8, and 1.0 and for the total number of peaks $N = 4$, 6, 8, 10, 50, and 100. There is good agreement between the expected values and the most probable values of the peaks for $\varepsilon = 1.0$, as seen from Tables 8 and 9. This confirms the validity of the above approximate expressions for finding the mean amplitudes of the peaks.

APPLICATION TO SINGLE-DEGREE-OF-FREEDOM OSCILLATOR

Preceding results have been applied to predict the relative peak amplitudes of the response of a viscously damped single-degree-of-freedom

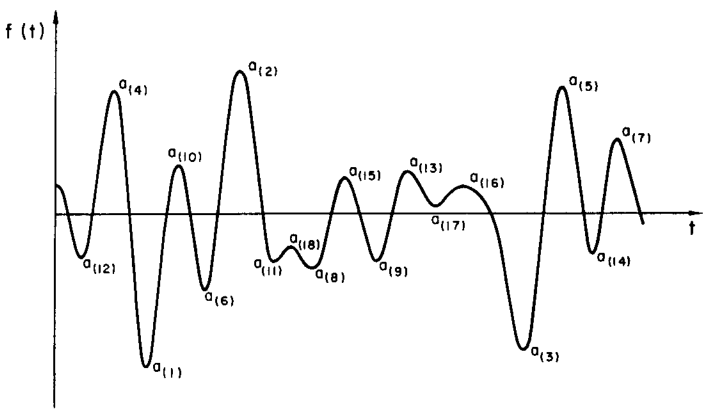

FIG. 1. Typical Example of Random Response Function, $f(t)$ with $a_{(1)}$, $a_{(2)}$, $a_{(3)}$, etc. as First-Order, Second-Order, Third-Order, etc., Peaks

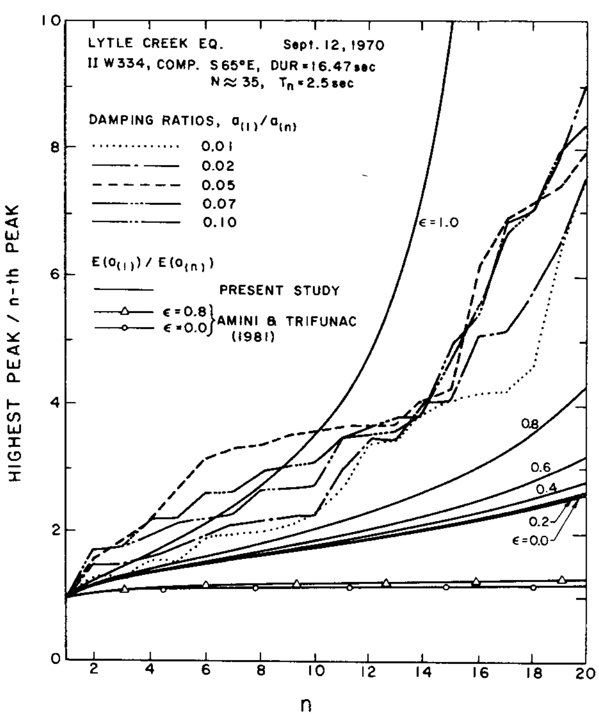

FIG. 2. Ratios $E[a_{(1)}]/E[a_{(n)}]$ from Present Study and Theory of Amini and Trifunac (1981) Compared with Ratio of Largest and n-th Peak, $a_{(1)}/a_{(n)}$, of Single-Degree-of-Freedom Oscillator Response to S65E Component of Lytle Creek Earthquake

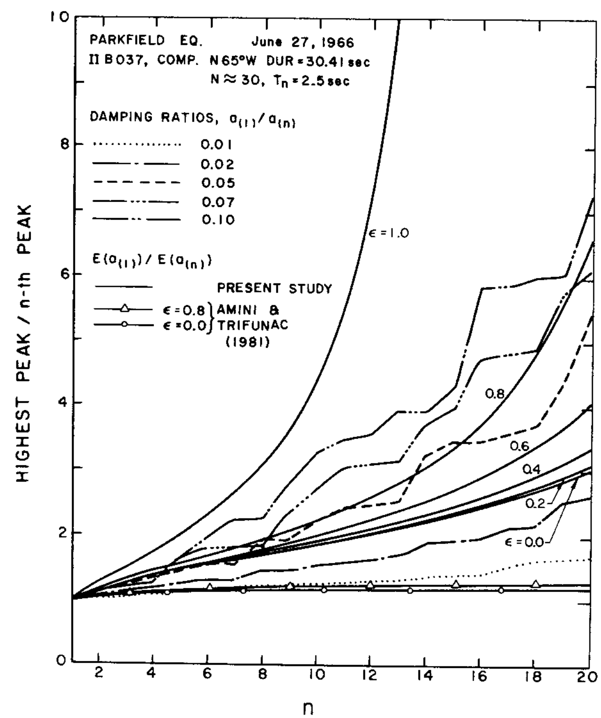

FIG. 3. Ratios $E[a_{(1)}/E(a_{(n)}]$ from Present Study and Theory of Amini and Trifunac (1981) Compared with Ratio of Largest and n-th Peak, $a_{(1)}/a_{(n)}$, of Single-Degree-of-Freedom Oscillator Response to N65W Component of Parkfield Earthquake

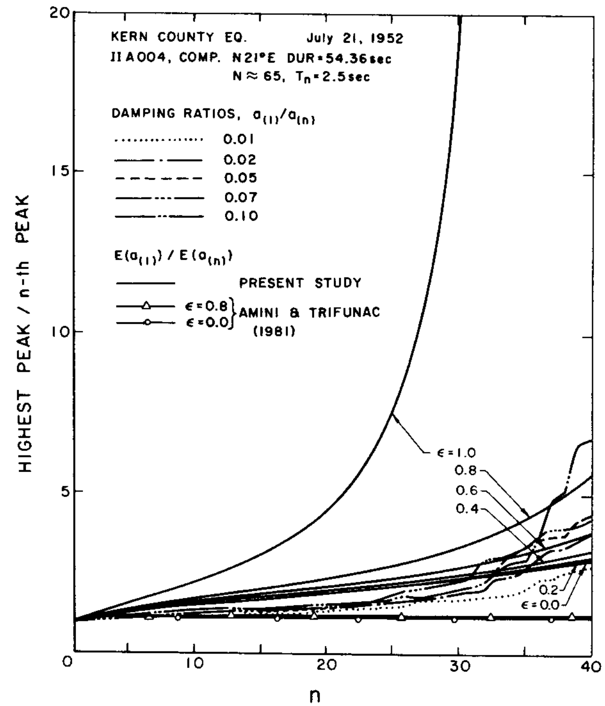

FIG. 4. Ratios $E[a_{(1)}]/E[a_{(n)}]$ from Present Study and Theory of Amini and Trifunac (1981) Compared with Ratio of Largest and n-th Peak, $a_{(1)}/a_{(n)}$, of Single-Degree-of-Freedom Oscillator Response to N21E Component of Kern County (Taft) Earthquake

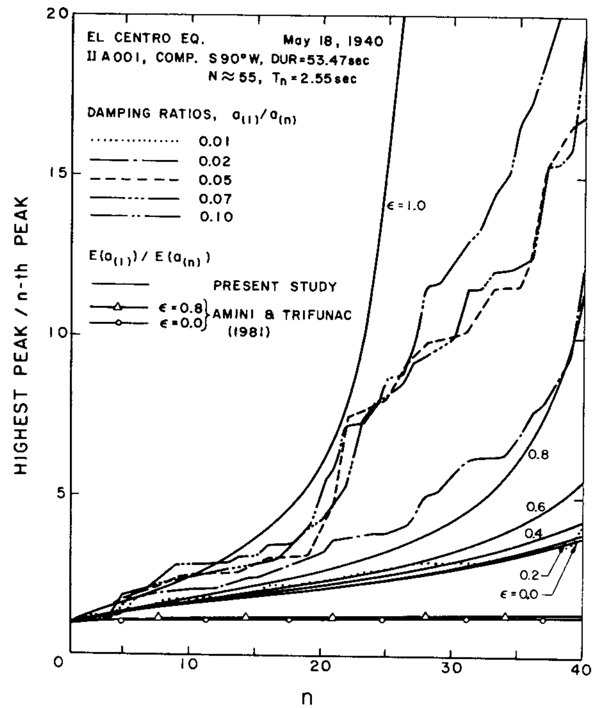

FIG. 5. Ratios $E[a_{(1)}]/E[a_{(n)}]$ from Present Study and Theory of Amini and Trifunac (1981) Compared with Ratio of Largest and n-th Peak, $a_{(1)}/a_{(n)}$, of Single-Degree-of-Freedom Oscillator Response to S90W Component of El Centro Earthquake

oscillator, excited by ground acceleration of strong motion earthquakes. Such an oscillator is described by the differential equation of motion

$$M\ddot{x}_r + C\dot{x}_r + Kx_r = -M\ddot{x}_g \quad \ldots \quad (50)$$

and its relative displacement amplitude at time t is given by the Duhamel integral for zero initial conditions

$$x_r(t) = \frac{-1}{\omega_n\sqrt{1-\zeta^2}} \int_0^t \ddot{x}_g(\tau)e^{-\zeta\omega_n(t-\tau)} \sin \omega_n\sqrt{1-\zeta^2}\,(t-\tau) \quad \ldots \quad (51)$$

where

$$\omega_n^2 = \frac{K}{M} \quad \ldots \quad (52)$$

is the square of natural frequency of the oscillator

$$\zeta = \frac{C}{2\sqrt{KM}} \quad \ldots \quad (53)$$

is the fraction of critical damping, and $\ddot{x}_g(t)$ is the input ground acceleration time history.

Response of the above oscillator has been considered for four different accelerograms as input excitation. These accelerograms are for the September 12, 1970, Lytle Creek Earthquake; June 27, 1966, Parkfield Earthquake; July 21, 1952, Kern County Earthquake; and May 18, 1940, El Centro Earthquake; and have been selected to produce varying degrees of departure from the assumptions of the stationarity and the mutual independence of the peaks of response.

Figs. 2–5 show the plots of the ratios of the highest- and the n-th-order peaks computed from the relative displacement response for values of critical damping ζ = 0.01, 0.02, 0.05, 0.07, and 0.10; and also the theoretical ratios of expected value of the highest- and the n-th-order peak. Similar comparisons for the most probable value of the highest- and the n-th-order peak lead to essentially the same results as those illustrated in Figs. 2 to 5. The theoretical ratios $E[a_{(1)}]/E[a_{(n)}]$ and $\mu[a_{(1)}]/\mu[a_{(n)}]$ have the same trend as the computed ratios $a_{(1)}/a_{(n)}$ for all values of n and for a wide range of damping values ζ. Though the assumptions of stationarity and independence of the maxima of response function $f(t)$ still continue, the order statistics has made remarkable improvement in the theoretical predictions, for large n, compared to the results obtained by Amini and Trifunac (1981), which are also plotted in these figures for the purpose of comparison.

Discussion

The results presented in this study have been derived under the assumptions of stationarity and mutual independence of the amplitudes of the maxima of random function $f(t)$. However, it has been observed that the theory is in good agreement with the actual computations.

As discussed by Amini and Trifunac (1981), results for $E[a_{(n)}]$ and $\mu[a_{(n)}]$ may be used to scale the relative response spectral amplitudes to get the

TABLE 10. Comparison of Proportion of Negative Peaks Computed from Equation 54 and that Obtained from Theory for N = 50

ε (1)	Number of negative peaks from theory (2)	Proportion of negative peaks from theory (3)	Proportion of negative peaks from Eq. 54 (4)
0.4	2	0.04	0.04
0.6	5	0.10	0.10
0.8	10	0.20	0.20
1.0	25	0.50	0.50

expected and the most probable values of spectral amplitudes that will be exceeded once, twice, thrice, etc., during the shaking corresponding to the unreduced spectrum. Results of Tables 1–7 may be used to find the spectral amplitudes that have specified probabilities of exceedance only once, twice, thrice, etc., during the shaking history of the structure.

As pointed out before, the success of the theory presented here is also supported by its ability to predict negative maxima. On physical grounds, the higher-order peaks should attain negative values as the value of parameter ε increases. Cartwright and Longuet-Higgins (1956) have shown that the proportion of negative maxima is given in terms of the parameter ε as

$$r = \frac{1}{2}[1 - (1 - \varepsilon^2)^{1/2}] \quad (54)$$

As shown in Table 10 for total number of peaks N = 50, predictions from Eq. 34 for expected value of peaks are in exact agreement with the results from Eq. 54. The same agreement has been found for other values of N and for the most probable values of the peaks. Thus, the order statistics makes the variation of the peak amplitudes more predictable on a physical basis. However, the negative peaks are not the result of the order statistics. Their origin lies in the probability density function $p(\eta)$ of peak amplitudes. As seen from analysis of $p(\eta)$, η may take on some negative values for nonzero values of ε. Variable η is a measure of height of peaks relative to the mean level of the function $f(t)$. Negative values of η simply mean that the corresponding maxima lie below the mean level.

Appendix I. References

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APPENDIX II. NOTATION

The following symbols are used in this paper:

$\bar{a}$	=	rms amplitude of peaks of $f(t)$;
a_n	=	n-th order peak of $f(t)$ when peaks are arranged in decreasing order of amplitudes;
a_{rms}	=	rms amplitude of $f(t)$;
c	=	viscous damping coefficient;
$E[a_{(n)}]$	=	expected value of $a_{(n)}$;
$E(\omega)$	=	energy density spectrum of $f(t)$;
$erf(x)$	=	error function;
$F_{(n)}(\eta)$	=	probability distribution function for n-th order peak;
$F(\omega)$	=	transform of $f(t)$;
$f_{(n)}(\eta)$	=	probability density function for n-th-order peak;
$f(t)$	=	random response function;
K	=	stiffness;
M	=	mass;
m_k	=	k-th order moment of $E_i(\omega)$;
N, N_T	=	total number of maxima (peaks) in response, $f(t)$;
$P(\eta)$	=	probability distribution function;
$p(\eta)$	=	probability density function;
r	=	proportion of negative maxima;
T	=	total duration of the response;
x_g	=	absolute ground displacement;
x_r	=	relative displacement of M;
ε	=	parameter describing the width of energy spectrum;
ε_{ij}	=	parameter ε for i-th story response due to j-th mode;
ζ	=	critical damping ratio;
$\eta(t)$, η	=	function $f(t)$ normalized by $(m_0)^{1/2}$;
$\mu[a_{(n)}]$	=	most probable value of $a_{(n)}$;
ϕ	=	phase; and
ω_n	=	circular frequency.