Least square model with spatial expansion: application to the inversion of earthquake source mechanism

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In a typical inversion of an earthquake source mechanism, using recorded strong motion data in the near field, the fault surface is subdivided into a rectangular mesh (Fig. 1) and the final dislocation amplitudes in ξ_1 and ξ_2 directions are found for each subfault by means of some inversion algorithm.

In this paper an alternative to the above approach is presented. We will describe the final dislocation amplitudes on the fault surface in terms of two-dimensional Fourier series and then, through inversion, we will compute the coefficients of these spatial harmonics. This leads to certain computational advantages and is helpful for understanding the spatial resolution of the dislocation amplitudes.

INTRODUCTION

So far most researchers have formulated the solution of the least square (LSQ) problem in earthquake source mechanism studies in terms of the unknown components of the dislocation vector[2,6,7]. Using the constraints and the regularization method, it is possible to estimate the dislocation vector at a discrete set of points on the fault (Fig. 1). However, the resolution of the small details of the dislocation amplitudes on the fault depends on the adopted number of subfaults. Therefore, to recover as much of the spatial details of the dislocation as possible, one is forced to use a dense set of subfaults. This will lead to an increase in the number of unknowns and the LSQ system of equations will become more unstable and ill-conditioned[2].

We note that for a typical case, the sampling rate Δt of the recorded strong motion is not a significant factor in the determination of the subfault sizes, instead, it is the computer storage and the numerical stability. In typical applications, the size of the subfaults is several kilometers by several kolometers. This does not yield good spatial resolution.

In general, a linear least square problem leads to the system of equations

$$A\vec{D}=\vec{f} \tag{1}$$

where A is $m\times n$ matrix, $\vec{D}$ is nth order unknown vector and $\vec{f}$ is mth order data vector. In the singular value decomposition of the resulting LSQ matrix A, by

$$U^T AV=S, \qquad A=USV^T \tag{2}$$

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where U is $m\times m$ orthogonal matrix and V is an $n\times m$ orthogonal matrix[3]. The columns of the orthogonal matrix V^T are linearly independent and form a base in the vector space where the unknown dislocation vector lies. Following an application of the Tikhonov regularization, the solution is then given by

$$\vec{D}=\sum_{n=1}^{K}\frac{\mu_n}{(1+\alpha^2\mu_n^2)}(\vec{f},\vec{u}_n)\vec{v}_n \tag{3}$$

where the triple $(\vec{v}_n, \vec{u}_n; \mu_n)$ is called the singular system for the matrix A[2], and α is the Tikhonov regularization parameter. For the sake of discussion we assume that the Tikhonov regularization parameter α is zero[4,5]. Then (3) takes the form:

$$\vec{D}=\sum_{n=1}^{K}\mu_n(\vec{f},\vec{u}_n)\vec{v}_n \tag{4}$$

From (4) it is seen that D belongs to the subspace spanned by the vectors $\{\vec{v}_n\}_1^K$. Then a question of resolution can be investigated by analyzing the linear combination (4) and the set of vectors $\{\vec{v}\}_1^K$.

Due to the Picard's theorem[1] the coefficients

$$|\mu_n(\vec{f},\vec{u}_n)|\to 0$$

as $n\to\infty$. Therefore, depending on the rate of convergence of $\mu_n(\vec{f}\vec{u}_n)$ we can have more or less significant contributions from the higher order eigenvectors $\vec{v}_n$. To maintain stability, very often small singular values μ_n are set to zero (truncated SVD) which means that we do not include some v_n in the expression (4), and consequently the resolution of $\vec{D}$ is reduced. For the time being we are concerned with the vectors $\{\vec{v}_n\}_1^K$

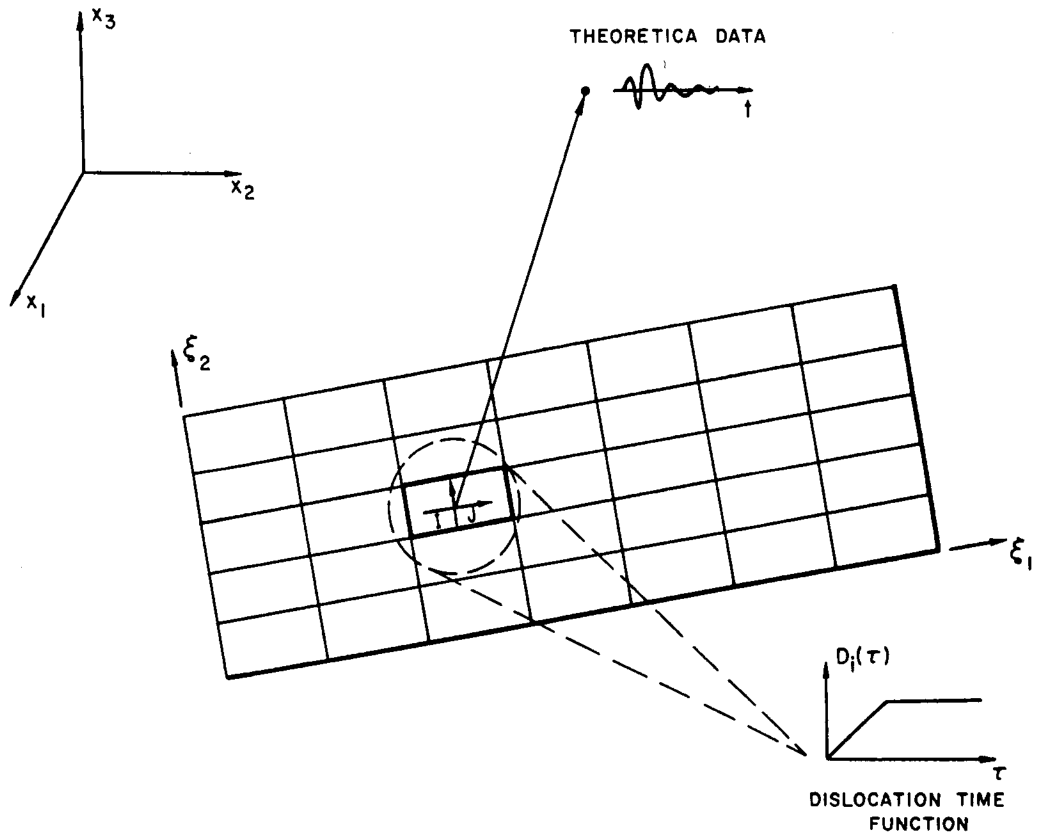

Fig. 1 Illustration of the subfaults, rupture function and theoretical data used to form the LSQ model

and we assume that the coefficients in (4) are well behaved.

The components of the $\bar{v}_n$ are of the same nature as $\vec{D}$ and represent dislocation values at some selected point of the fault surface, usually centre of the subfault. According to the formulation of Jordanovski *et al.*[2] we may take the odd components of $\bar{v}_n$ to represent dislocations in the ξ_1-direction $D_1(i,j)$, while the even components then represent dislocations in the ξ_2-direction $_2(i,j)$ (Fig. 1). Consequently, the number of $\bar{v}_n$ vectors is limited by the number of subfaults. The calculated values of $D_1(i,j)$ or $D_2(i,j)$ represent the final offsets of the ramp function of the (i,j)th subfault and represent the average values of the dislocations $D_1(\xi_1, \xi_2)$ or $D_2(\xi_1, \xi_2)$ over the subfault. Hence, if the subfault has significant dimensions then averaging can introduce significant error. One way to avoid this error is to make fine subdivision of the fault, but then the number of unknowns will increase, and also due to growing instability one may be forced to reject some small singular values. This results in taking only first k vectors in the expression (4). Therefore, there is some upper index n_0 such that we cannot obtain $\bar{v}_n$ for $n > n_0$. To extend the resolution we will use the following approach.

Least square model with spatial expansion

Since the dislocation components $D_1(\xi_1, \xi_2)$ and $D_2(\xi_1, \xi_2)$ are defined over the two-dimensional fault plane and are elements of the real vector space $L_2(\Sigma)$, one can try to find a set of orthogonal base functions $\{\psi_{ij}(\xi_1, \xi_2)\} \varepsilon L_2(\Sigma)$, and express $D_1(\xi_1, \xi_2)$ and $D_2(\xi_1, \xi_2)$ as

$$D(\xi_1, \xi_2) = \sum_{i=1}^{\bar{N}} \sum_{j=1}^{\bar{M}} d_{ij}\psi_{ij}(\xi_1, \xi_2) \tag{5}$$

where

$$d_{ij} = \langle D(\xi_1, \xi_2), \psi_{ij}(\xi_1, \xi_2)\rangle = \int_0^L \int_0^W D(\xi_1, \xi_2)\psi_{ij}(\xi_1, \xi_2) d\xi_1 d\xi_2 \tag{6}$$

using the orthonormal properties of the ψ_{ij}'s. Here L and W are the length and the width of the fault. The orthogonality conditions require that:

$$\langle \psi_{ij}, \psi_{kl} \rangle = \delta_{ij}\delta_{jl} \tag{7}$$

Since the choice of $\psi_{ij}(\xi_1, \xi_2)$ is arbitrary, one can choose them to satisfy the dislocation conditions on the fault boundaries. Thus ψ_{ij} have to vanish at the edges of the fault. If the fault breaks the surface then we can replace the zero displacement by the zero stress boundary conditions for the surface part of the fault.

It seems natural to choose the sine and cosine functions. Besides, the harmonic expansion has certain familiar meaning when we discuss the resolution. Therefore, for deep faults ($h > 0$ in Fig. 2) we may use

$$\psi_{ij}(\xi_1, \xi_2) = \frac{2}{\sqrt{LW}} \sin\frac{i\pi}{L}\xi_1 \sin\frac{j\pi}{H}\xi_2 \tag{8}$$

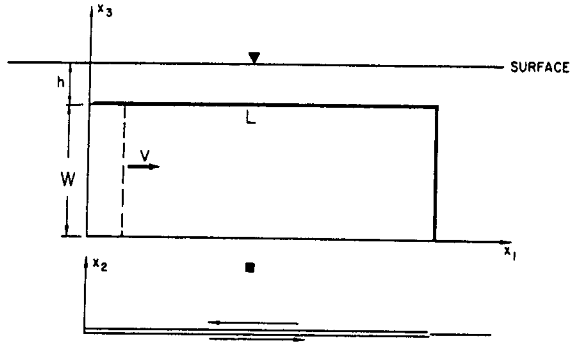

Fig. 2. Spatial position of the station with respect to the fault

and for the surface fault ($h=0$ in Fig. 2)

$$\psi_{ij}(\xi_1,\xi_2)=\frac{2}{\sqrt{LW}}\sin\frac{i\pi}{L}\xi_1\sin\frac{j\pi}{2H}\xi_2 \tag{9}$$

Note that in both cases the applicable boundary conditions are satisfied. Also the condition (7) is satisfied. Then the coefficients d_{ij} are given by (6).

To incorporate the expansion of (5) into the LSQ model two procedures can be used. In the first procedure the LSQ system

$$A\vec{D}=\vec{f} \tag{10}$$

is obtained by employing the theoretical displacements from each subfault (Fig. 1). In the next step one can introduce new unknowns d_{ij} through expansion (6). Following the ordering mapping, such that for $K=1, 2, 3, \ldots, J$, where $J=M\cdot N$

$$\begin{aligned} D_{2K-1}&=D_1(i\Delta\xi_1, j\Delta\xi_2)\\ D_{2K}&=D_2(i\Delta\xi_1, j\Delta\xi_2)\end{aligned} \tag{11}$$

where K is defined by

$$K=(i-1)M+j; \qquad i=1, N \quad j=1, M \tag{12}$$

and N and M represent the number of fault segments in ξ_1 and ξ_2 directions. The set of values $D(i\Delta\xi_1, j\Delta\xi_2)$ are transformed into one dimensional unknown vector $\{D_k\}_1^{2J}$, where 'J' is the number of subfaults. Here $(i\Delta\xi_1, j\Delta\xi_2)$ is the coordinate of the centre of the (i,j)th subfault element, with $\Delta\xi_1$ by $\Delta\xi_2$ being the size of each subfault element (Fig. 1). The relationship between $\vec{D}$ and the coefficients $\{d_{ij}\}$ can be expressed in a matrix form

$$\vec{D}=H\vec{d} \tag{13}$$

where H is $(2J)\times(\bar{M}\cdot\bar{N})$ matrix and its elements are the values of the functions $\psi_{ij}(i\Delta\xi_1, j\Delta\xi_2)$, and $\vec{d}$ is an $(\bar{M}\cdot\bar{N})$-vector.

If we substitute $\vec{D}$ from (13) into (10), the new LSQ problem in terms of the unknown vector $\vec{d}$ will be

$$AH\vec{d}=\vec{f} \tag{14}$$

or after singular-value decomposition of A

$$USV^THd=\vec{f} \tag{15}$$

or

$$USK^T\vec{d}=\vec{f} \tag{16}$$

with

$$K=H^TV \tag{17}$$

and if there are constraints

$$G\vec{D}\geqslant\vec{h} \tag{18}$$

the new constraints involving the vector $\vec{d}$ are

$$GH\vec{d}\geqslant\vec{h} \tag{19}$$

Therefore, the new LSQ system is now given by (16) and (19). It can be shown that

$$H^TH=I \tag{20}$$

and that the elements in the *i-th* column of the matrix K, are the coefficients of the *i-th* column of the matrix V with respect to the base. Since the new base functions are harmonic spatial functions, one can gain better knowledge and understanding of the resolution level.

In general, K^T is not a square matrix and depends on the matrix H, or on the number of coefficients taken in the expansion (5). If one chooses to consider only the first few harmonic functions, then the number of columns can be less than the number of rows. This will certainly reduce the number of unknowns, but unless we know that the dislocation is smooth enough, this will also reduce the accuracy.

Theoretically, it seems that the number of harmonics taken in the expansion (5) can be as large as one wishes but due to the sampling theorem, the highest spatial harmonic is limited to

$$\frac{\bar{N}}{L}=\frac{1}{2\Delta\xi_1}; \qquad \frac{\bar{M}}{W}=\frac{1}{2\Delta\xi_2} \tag{21}$$

or

$$\bar{N}=\frac{N}{2} \quad\text{and}\quad \bar{M}=\frac{M}{2} \tag{22}$$

where N is the number of subfaults in the ξ_1 direction and similarly, M is the number of subfaults in the ξ_2 direction. Therefore, with this approach there appear to exist similar limitations as with the original LSQ problem. Yet, there are two advantageous points that should be emphasized. Firstly, for the same resolution as in the original LSQ model, one now requires only one half the unknowns in the LSQ model. Secondly, once the coefficients $\{d_{ij}\}$ are calculated, the dislocations $D_1(\xi_1,\xi_2)$ and $D_2(\xi_1,\xi_2)$ are known as a continuous function over the fault plane.

In the second approach, which avoids the representation in terms of fault segments, we consider each harmonic function as a final offset of the ramp time

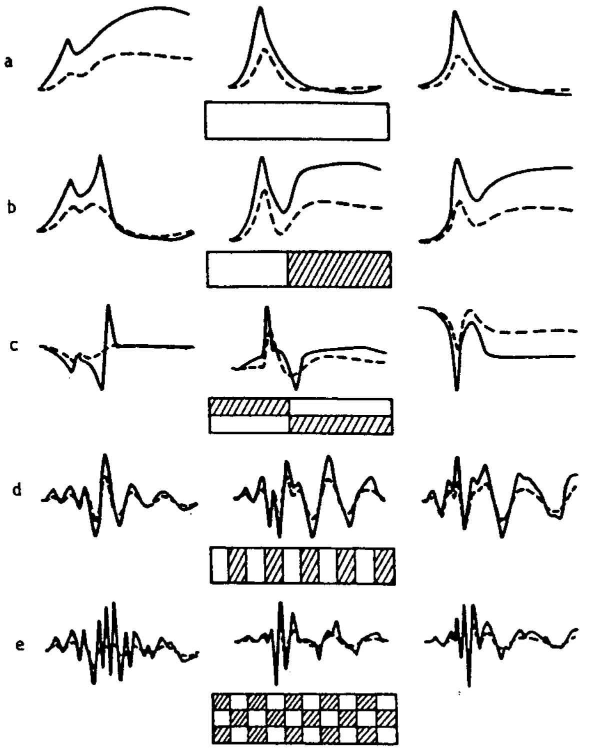

Fig. 3 Illustration of the differences between ramp (solid lines) and sine shape of the fault dislocation (dashed lines). The surface displacements (square in Fig. 2) are shown for the displacements in X_1 (left), X_2 (centre) and X_3 (right) directions

function and calculate the theoretical displacement due to this type of dislocation. Therefore, one can write that

$$D(\xi_1, \xi_2) = \psi_{ij}(\xi_1, \xi_2) \tag{23}$$

or a dislocation function is of the form:

$$a(\xi_1, \xi_2, t) = \begin{cases} \dfrac{\psi_{ij}(\xi_1, \xi_2)}{T} t & 0 < t < T \\ \psi_{ij}(\xi_1, \xi_2) & T < t \end{cases} \tag{24}$$

Justification for this step can be sought in the fact that final offset of the ramp function does not depend explicitly on time. This means that at a given point, the dislocation grows as a ramp function in time until it reaches the value determined by $\psi_{ij}(\xi_1, \xi_2)$ at that point.

Example

A set of computer programs has been written for this case. Test examples have been compared with the results from the previous programs[2] which used constant ramp functions. In Fig. 3 these results are illustrated where the two solutions are compared. The solid line represents the constant unit ramp dislocation while the dashed line is a dislocation given by (24). Different approximations for the final offset of the ramp functions are presented here.

For example, in Fig. 3(a), to model constant dislocation over the entire fault we used only the first harmonic in the expansion, i.e. a half-sine in the ξ_1-direction and a half-sine in the ξ_2-direction and with unit amplitude. Obviously one cannot expect to have the same values of the displacements, only the waveforms are expected to be preserved. This is the case for all three components of a station (square) of Fig. 3. As shown in Fig. 3(b), we took $D(\xi_1, \xi_2)$ to be one $(+1)$ over the first half of the fault (plain area) and negative one (-1) over the second half of the fault (shaded area). This is modelled by the full sine in the ξ_1-direction and by a half sine in the ξ_2-direction. Again, the waveform is well preserved. Following the convention that the shaded areas have negative dislocation (equal to -1) and the plain areas have positive dislocation (equal to 1), one can peruse the rest of Fig. 2. As it might be expected for higher resolutions, the two models are in good agreement due to the fact that the higher order sine function can be approximated successfully by the rectangular wave form.

It appears that this approach does not limit the higher spatial resolution. The only difficulty is in the required computational time as well as in the ill-conditioned nature of the resulting LSQ system. The resulting inverse procedure can be summarized by the following steps:

(I) Calculate the response at a given site due to each dislocation $\psi_{ij}(\xi_1, \xi_2)$ by setting the corresponding coefficient to one.

(II) Form the least squares system with $\vec{d}$ as unknown from:

$$A\vec{d} = \vec{f} \tag{25}$$

(III) After solving (25), calculate the dislocation at a given point using expansion (5), written in the matrix form,

$$\vec{D} = H\vec{d} \tag{26}$$

(IV) If there are constraints then with (26) express them in terms of d as

$$GH\vec{d} \geqslant \vec{h} \tag{27}$$

Note that the representation of the dislocation through the series (5) is especially convenient for any linear constraints. For example, if the *p-th* order derivative is used for the regularization, then we are able to obtain the exact *p-th* derivative instead of its discrete counterpart.

For a detailed example of how this approach can be implemented in the study of the source mechanism of a particular earthquake, we refer the reader to Chapter V of Jordanovski *et al.*[2].

CONCLUSIONS

In search of the fine details in the spatial variations of the dislocation amplitudes over a fault surface, one may proceed by making progressively smaller subdivision of the fault (Fig. 1), i.e. make $\Delta\xi_1$ and $\Delta\xi_2$ progressively smaller. However, then the number of the unknowns will increase, and due to the growing instability of the generalized inverse or in the method of Tikhonov regularization, one may be forced to reject some small singular values. This results in taking only the first K vectors in expansion (3) where $K < N \cdot M$.

In this paper we have presented a natural alternative which consists of expanding the dislocation amplitudes over the fault surface into a Fourier series involving sines and cosines. The advantages of this approach are that the number of harmonics is limited only by the sampling theorem, that for the same resolution as in the original LSQ model one now requires only one half the unknowns and that the dislocation amplitudes are now known as continuous functions over the fault plane. The disadvantage of this approach lies in the considerable computational time.

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